Noncooperative Oligopoly in Markets with a Cobb-Douglas Continuum of Traders

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Abstract

In this paper, we reconsider two models of noncooperative oligopoly in general equilibrium proposed by Busetto et al. ((2008), (2011)): a version of the Shapley’s window model for mixed exchange economies à la Shitovitz and its reformulation à la Cournot-Walras. We introduce the assumption that preferences of the traders belonging to the atomless part are represented by Cobb-Douglas utility functions. This assumption permits us to prove the existence of a Cournot-Nash equilibrium of the Shapley’s window model - called Cobb-Douglas-Cournot-Nash equilibrium - without introducing further assumptions on atoms’ endowments and preferences previously used by Busetto et al. (2011). Then, we show that the set of the Cobb-Douglas-Cournot-Nash equilibrium allocations coincides with the set of the Cournot-Walras equilibrium allocations.

Keywords: strategic market games, noncooperative oligopoly, atoms, atomless part.

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1 Introduction

Noncooperative oligopoly in interrelated markets has been modeled within two main approaches. The first is the strategic market game approach, initiated by Shapley and Shubik (see also Dubey and Shubik (1978), Postlewaite and Schmeidler (1978), Okuno et al. (1980), Mas-Colell (1982), Sahi and Yao (1989), Amir et al. (1990), Peck et al. (1992), Dubey and Shapley (1994), among others). In this class of models, all traders behave strategically and prices are determined according to non-Walrasian pricing rules. The second is the Cournot-Walras approach, initiated by Gabszewicz and Vial (1972) for economies with production (see also Roberts and Sonnenschein (1977), Roberts (1980), Mas-Colell (1982), Dierker and Grodal (1986), among others), and by Codognato and Gabszewicz (1991) for pure exchange economies (see also Codognato and Gabszewicz (1993), d’Aspremont et al. (1997), Gabszewicz and Michel (1997), Shitovitz (1997), Julien and Tricou (2005), (2009), among others). In this class of models, some agents behave strategically while others behave competitively and prices are determined according to the Walrasian pricing rule. Strategic agents determine their strategies as in the Cournot game (see Cournot (1838)) taking into account the Walrasian price correspondence. Both classes of models aim at studying the working and the consequences of market power in a general equilibrium framework.

More recently, Busetto et al. (2008), (2011) introduced two models of noncooperative oligopoly in general equilibrium, inspired by a strategic market game originally proposed by Lloyd S. Shapley and known as Shapley’s window model. This model was further analyzed by Sahi and Yao (1989) in exchange economies with a finite number of traders, and Codognato and Ghosal (2000) in exchange economies with an atomless continuum of traders. More precisely, Busetto et al. (2011), taking inspiration from a seminal paper by Okuno et al. (1980), proved the existence of a Cournot-Nash equilibrium of the Shapley’s window model associated with an exchange economy à la Shitovitz (1973), i.e., with atoms and an atomless part. Instead, Busetto et al. (2008) provided a respecification à la Cournot-Walras of this mixed exchange economy assuming that atoms behave à la Cournot while the atomless part behaves à la Walras. They showed, by an example, that the set of the Cournot-Nash equilibrium allocations does not coincide with the set of the Cournot-Walras equilibrium allocations in a one-stage setting. Nevertheless, they introduced a reformulation of the Shapley’s model as a two-stage game, where the atoms move in the first stage and the atomless sector moves in the second stage, and showed that the set of the Cournot-Walras equilib-
rium allocations coincides with a specific set of subgame perfect equilibrium allocations of this two-stage game.

In this paper, we reconsider these two models under the assumption that preferences of the traders belonging to the atomless part are represented by Cobb-Douglas utility functions. Beyond their tractability to compute solutions in theoretical models, Cobb-Douglas utility functions will turn out very useful to establish the relationships among the equilibrium concepts studied in this paper.

We first show the existence of a Cobb-Douglas-Cournot-Nash equilibrium - i.e., a Cournot-Nash equilibrium where the strategies of the traders belonging to the atomless part depend on the parameters of their Cobb-Douglas utility functions - under the following further assumptions: (i) each trader is endowed with a strictly positive amount of at least one commodity and each commodity is held, in the aggregate, by the atomless part; (ii) atoms’ utility functions are continuous, strongly monotone, and quasi-concave; (iii) traders’ utility functions are jointly measurable. Busetto et al. (2011) proved the existence of a more general Cournot-Nash equilibrium by using less restrictive assumptions on the atomless part’s endowments and preferences. In particular, they assumed that the atomless part has continuous, strongly monotone, and quasi-concave preferences without requiring that it holds, in the aggregate, each commodity. Nevertheless, our proof is not merely a special case of theirs: indeed, following Sahi and Yao (1989), they imposed the hypothesis that there exists at least two atoms with endowments and indifference curves contained in the strict interior of the commodity space. This hypothesis can be removed in our context, therefore, our proof allows us to deal with cases where all atoms have corner endowments and indifference curves which cross the boundary of the commodity space.

Then, following Busetto et al. (2008), we provide a respecification à la Cournot-Walras of our model and we prove that, under the assumptions (i)-(iii) mentioned above, the set of the Cobb-Douglas-Cournot-Nash equilibrium allocations coincides with the set of the Cournot-Walras equilibrium allocations. This result contrasts with the example provided by Busetto et al. (2008) mentioned above which shows that it may not hold if preferences of the traders belonging to the atomless part are not represented by Cobb-Douglas utility functions.

The paper is organized as follows. In Section 2, we present the mathematical model. In Section 3, we show the existence of the Cobb-Douglas-Cournot-Nash equilibrium. Section 4 is devoted to the Cournot-Walras equilibrium. Section 5 aims at studying the relationship between the Cobb-
Douglas-Cournot-Nash and the Cournot-Walras equilibrium. Section 6 concludes.

2 The mathematical model

We consider a pure exchange economy, \( E \), with large traders, represented as atoms, and small traders, represented by an atomless part. The space of traders is denoted by the measure space \((T, \mathcal{T}, \mu)\), where \( T \) is the set of traders, \( \mathcal{T} \) is the \( \sigma \)-algebra of all \( \mu \)-measurable subsets of \( T \), and \( \mu \) is a real valued, non-negative, countably additive measure defined on \( \mathcal{T} \). We assume that \((T, \mathcal{T}, \mu)\) is finite, i.e., \( \mu(T) < \infty \). This implies that the measure space \((T, \mathcal{T}, \mu)\) contains at most countably many atoms. Let \( T_1 \) denote the set of atoms and \( T_0 = T \setminus T_1 \) the atomless part of \( T \). A null set of traders is a set of measure 0. Null sets of traders are systematically ignored throughout the paper. Thus, a statement asserted for “each” trader in a certain set is to be understood to hold for all such traders except possibly for a null set of traders. The word “integrable” is to be understood in the sense of Lebesgue.

There are \( l \) different commodities. A commodity bundle is a point in \( \mathbb{R}^l_+ \). An assignment (of commodity bundles to traders) is an integrable function \( x: T \rightarrow \mathbb{R}^l_+ \). There is a fixed initial assignment \( w \), satisfying the following assumption.

**Assumption 1.** \( w(t) > 0 \), for each \( t \in T \), \( \int_{T_0} w(t) \, d\mu \gg 0 \).

Furthermore, as in Sahi and Yao (1989), we can assume, for convenience, that \( \int_T w^j(t) \, d\mu = 1, \, j = 1, \ldots, l \). An allocation is an assignment \( x \) for which \( \int_T x(t) \, d\mu = \int_T w(t) \, d\mu \).

The preferences of each trader \( t \in T \) are described by a utility function \( u_t: \mathbb{R}^l_+ \rightarrow \mathbb{R} \). We assume that atoms’ preferences are represented by utility functions which satisfy standard continuity, monotonicity, and concavity assumptions. We do not assume, as Busetto et al. (2011) did, that there exist atoms whose indifference curves lie in the strict interior of the commodity space. On the other hand, we introduce the crucial assumption that the traders belonging to the atomless part have Cobb-Douglas utility functions.

**Assumption 2.** \( u_t: \mathbb{R}^l_+ \rightarrow \mathbb{R} \) is continuous, strongly monotone, and quasi-concave, for each \( t \in T_1 \), \( u_t(x) = x^t \alpha^1(t) \cdots x^t \alpha^l(t) \), for each \( t \in T_0 \) and for each \( x \in \mathbb{R}^l_+ \), where \( \alpha: T_0 \rightarrow \mathbb{R}^l_+ \) is a function such that \( \alpha^j(t) > 0 \), \( j = 1, \ldots, l \), \( \sum_{j=1}^l \alpha^j(t) = 1 \), for each \( t \in T_0 \).
Let $\mathcal{B}(R^l_+)$ denote the Borel $\sigma$-algebra of $R^l_+$. Moreover, let $\mathcal{T} \otimes \mathcal{B}$ denote the $\sigma$-algebra generated by the sets $E \times F$ such that $E \in \mathcal{T}$ and $F \in \mathcal{B}$.

**Assumption 3.** $u : T \times R^l_+ \rightarrow R$ given by $u(t, x) = u_t(x)$, for each $t \in T$ and for each $x \in R^l_+$, is $\mathcal{T} \otimes \mathcal{B}$-measurable.

A price vector is a vector $p \in R^l_+$. We define, for each $p \in R^l_+$, a correspondence $\Delta_p : T \rightarrow \mathcal{P}(R^l)$ such that, for each $t \in T$, $\Delta_p(t) = \{x \in R^l_+ : px = pw(t)\}$, a correspondence $\Psi_p : T \rightarrow \mathcal{P}(R^l)$ such that, for each $t \in T$, $\Psi_p(t) = \{x \in R^l_+ : \text{for all } y \in \Delta_p(t), u_t(x) \geq u_t(y)\}$, and finally a correspondence $X_p : T \rightarrow \mathcal{P}(R^l)$ such that, for each $t \in T$, $X_p(t) = \Delta_p(t) \cap \Psi_p(t)$.

A Walras equilibrium of $E$ is a pair $(p^*, x^*)$, consisting of a price vector $p^*$ and an allocation $x^*$, such that $x^*(t) \in X_p(t)$, for each $t \in T$. By Assumption 2, for each $p \in R^l_{++}$, it is possible to define the atomless part’s Walrasian demands as a function $x^0(\cdot, p) : T_0 \rightarrow R^l_+$ such that $x^0(t, p) = X_p$, for each $t \in T_0$. It is immediate to verify that $x^0(t, p) = \frac{\alpha(t)}{p} \sum_{i=1}^l p^i w_i(t)$, $j = 1, \ldots, l$, for each $t \in T_0$. The following proposition shows that this function is integrable.

**Proposition 1.** Under Assumptions 1, 2, and 3, the function $x^0(\cdot, p)$ is integrable, for each $p \in R^l_{++}$. 

**Proof.** Let $p \in R^l_{++}$. The restriction of $w$ to $T_0$ is integrable as $w$ is integrable. Now, we prove that $\alpha$ is a measurable function. Consider a commodity bundle $y \in R^l_+$. Let $u^0(\cdot, y)$ denote the restriction of $u(\cdot, y)$ to $T_0$. The function $u(\cdot, y)$ must be measurable as, by Assumption 3, $u(\cdot, \cdot)$ is $\mathcal{T} \otimes \mathcal{B}$-measurable (see Theorem 4.48 in Aliprantis and Border (2006), p. 152). Then, the function $u^0(\cdot, y)$ is also measurable. Suppose that $\alpha$ is not measurable. Then, there is an open set $O \in R^l_+$ such that $\alpha^{-1}(O)$ is not a $\mu$-measurable set. Let $f : R^l_+ \rightarrow R^l_+$ be a function such that $f(v) = (y_1^v, \ldots, y_l^v)$, for each $v \in R^l_+$. $f(O)$ is an open set as $f$ is a homeomorphism. Suppose that $\tau \in \alpha^{-1}(O)$. Then, $f(\alpha(\tau)) \in f(O)$. But then, $\tau \in (u^0(\cdot, y))^{-1}(f(O))$. Therefore, $\alpha^{-1}(O) \subseteq (u^0(\cdot, y))^{-1}(f(O))$. Suppose that $\tau \in (u^0(\cdot, y))^{-1}(f(O))$. Moreover, suppose that $\tau \notin \alpha^{-1}(O)$. Then, $\alpha(\tau) \notin O$. But then, $u^0(\tau, y) \notin f(O)$, a contradiction. Therefore, $\alpha^{-1}(O) \subseteq (u^0(\cdot, y))^{-1}(f(O))$. Then, $\alpha^{-1}(O) = (u^0(\cdot, y))^{-1}(f(O))$. But then, $\alpha^{-1}(O)$ is $\mu$-measurable as $u^0(\cdot, y)$ is measurable, a contradiction. Therefore, $\alpha$ is measurable. Hence, $x^0(\cdot, p)$ is integrable as it is measurable and $x^0(t, p) < \frac{\sum_{i=1}^l p^i w_i(t)}{p^i}, j = 1, \ldots, l$, for each $t \in T_0$. $\blacksquare$
3 Cobb-Douglas-Cournot-Nash equilibrium

We introduce now the strategic market game, $\Gamma$, associated with $E$, following the reformulation of the Shapley’s window model proposed by Busetto et al. (2011). Let $b \in R_+^2$ be a vector such that $b = (b_{11}, b_{12}, \ldots, b_{l-1}, b_{ll})$. A strategy correspondence is a correspondence $B : T \rightarrow P(R_+^2)$ such that, for each $t \in T$, $B(t) = \{b \in R_+^2 : \sum_{j=1}^l b_{ij} \leq w^i(t), i = 1, \ldots, l\}$. A strategy selection is an integrable function $b : T \rightarrow R_+^2$, such that, for each $t \in T$, $b(t) \in B(t)$. For each $t \in T$, $b_{ij}(t)$, $i, j = 1, \ldots, l$, represents the amount of commodity $i$ that trader $t$ offers in exchange for commodity $j$. Given a strategy selection $b$, we define the aggregate matrix $\bar{B} = (\int_T b_{ij}(t) d\mu)$. Moreover, we denote by $b \setminus b(t)$ a strategy selection obtained by replacing $b(t)$ in $b$ with $b \in B(t)$. With a slight abuse of notation, $b \setminus b(t)$ will also represent the value of the strategy selection $b \setminus b(t)$ at $t$.

Then, we introduce two further definitions (see Sahi and Yao (1989)).

Definition 1. A nonnegative square matrix $A$ is said to be irreducible if, for every pair $(i, j)$, with $i \neq j$, there is a positive integer $k = k(i, j)$ such that $a^{(k)}_{ij} > 0$, where $a^{(k)}_{ij}$ denotes the $ij$-th entry of the $k$-th power $A^k$ of $A$.

Definition 2. Given a strategy selection $b$, a price vector $p$ is market clearing if

$$p \in R_+^l, \sum_{i=1}^l p^i \bar{b}_{ij} = p^j(\sum_{i=1}^l \bar{b}_{ji}), j = 1, \ldots, l. \quad (1)$$

By Lemma 1 in Sahi and Yao (1989), there is a unique, up to a scalar multiple, price vector $p$ satisfying (1) if and only if $\bar{B}$ is irreducible. Then, we denote by $p(b)$ a function which associates with each strategy selection $b$ the unique, up to a scalar multiple, price vector $p$ satisfying (1), if $\bar{B}$ is irreducible, and is equal to 0, otherwise.

Given a strategy selection $b$ and a price vector $p$, consider the assignment determined as follows:

$$x^j(t, b(t), p) = w^j(t) - \sum_{i=1}^l b_{ji}(t) + \sum_{i=1}^l b_{ij}(t)p_i \frac{p_j}{p^j}, \text{ if } p \in R_+^l,$$

$$x^j(t, b(t), p) = w^j(t), \text{ otherwise},$$

$j = 1, \ldots, l$, for each $t \in T$. 

6
According to this rule, given a strategy selection $b$ and the function $p(b)$, the traders’ final holdings are determined as follows:

$$x(t) = x(t, b(t), p(b)),$$

for each $t \in T$. It is straightforward to show that the assignment corresponding to the final holdings is an allocation.

This reformulation of the Shapley’s window model for mixed exchange economies allows us to define the following concept of Cournot-Nash equilibrium.

**Definition 3.** A strategy selection $\hat{b}$ such that $\bar{\hat{B}}$ is irreducible is a Cournot-Nash equilibrium of $\Gamma$ if

$$u_t(x(t, \hat{b}(t), p(\hat{b}))) \geq u_t(x(t, \hat{b}(t), p(\hat{b} \setminus b(t)))),$$

for each $b \in B(t)$ and for each $t \in T$.

In order to define the notion of Cobb-Douglas-Cournot-Nash equilibrium we consider a function $b^0 : T_0 \rightarrow R^l_+$ such that $b^0_{ij}(t) = w^i(t)\alpha^j(t)$, $i, j = 1, \ldots, l$, for each $t \in T_0$. Then, we have $b^0(t) \in B(t)$, for each $t \in T_0$, as $w^i(t)\alpha^j(t) \geq 0$, $i, j = 1, \ldots, l$, and $\sum_{j=1}^l w^i(t)\alpha^j(t) = w^i(t)$, $i = 1, \ldots, l$, for each $t \in T_0$. The following proposition shows that the function $b^0$ is integrable.

**Proposition 2.** Under Assumptions 1, 2, and 3, the function $b^0$ is integrable.

**Proof.** $b^0$ is measurable as the restriction of $w$ to $T_0$ is measurable and we know, from the proof of Proposition 1, that $\alpha$ is measurable. Then, $b^0$ is integrable as $b^0_{ij}(t) < w^i(t)$, $i, j = 1, \ldots, l$, for each $t \in T_0$. $\blacksquare$

We can now provide the definition of a Cobb-Douglas-Cournot-Nash equilibrium of $\Gamma$.

**Definition 4.** A strategy selection $\hat{b}$ is a Cobb-Douglas-Cournot-Nash equilibrium of $\Gamma$ if it is a Cournot-Nash equilibrium of $\Gamma$ and $\hat{b}(t) = b^0(t)$, for each $t \in T_0$.

The rest of this section is devoted to prove the existence of a Cobb-Douglas-Cournot-Nash equilibrium. To this end, we first need to introduce an auxiliary game, which we call $\Gamma_1$, where only the atoms act strategically, taking $b^0$ as given. We will also need to show that Lemmas 3 and 4 in Sahi and Yao (1989) can readapted to our framework. The game $\Gamma_1$ has, *mutatis
mutandis, the same structure as $\Gamma$. We can now establish a relationship between $\Gamma_1$ and $\Gamma$ which will be used in the proof of the existence theorem. Let $b^1 : T_1 \rightarrow R^n_+$ be a function such that $b^1(t) \in B(t)$, for each $t \in T_1$. $b^1$ is integrable as $\sum_{t \in T_1} \int_t b^1(t) d\mu \leq \sum_{t \in T_1} \int_t w(t) d\mu = \int_{T_1} w(t) d\mu < \infty$. Then, $b^1$ is a strategy selection of $\Gamma_1$. Given a strategy selection $b^1$ of $\Gamma_1$, let $b^{10} : T \rightarrow R^n_+$ be a function such that $b^{10}(t) = b^1(t)$, for each $t \in T_1$, and $b^{10}(t) = b_0^0(t)$, for each $t \in T_0$. Then, $b^{10}$ is a strategy selection of $\Gamma$ as $\int_{T_1} b^1(t) d\mu + \int_{T_0} b^0(t) d\mu \leq \int_{T_1} w(t) d\mu + \int_{T_0} w(t) d\mu = \int_T w(t) d\mu < \infty$. Consider an atom $\tau \in T_1$. Given a strategy selection $b^{10}$, consider a vector $\hat{\beta} \in B(\tau)$. Suppose that $\hat{b}_{ij} \neq b^{10}_{ij}(\tau)$, for at least a pair $(i, i)$, and $\hat{b}_{ij} = b^{10}_{ij}(\tau)$, for the remaining pairs $(i, j)$. Then, it is straightforward to verify that $p(b^{10}) = p(b^{10} \setminus \hat{b}(\tau))$. Therefore, as in Sahi and Yao (1989), we can assume, for convenience and without loss of generality, that, given a strategy selection $b^{10}$, $\sum_{j=1}^l b^{10}_{ij}(t) = w^i(t)$, $i = 1, \ldots, l$, for each $t \in T_1$. Then, given a strategy selection $b^{10}$, the corresponding aggregate matrix $\bar{B}^{10}$ is row-stochastic. Moreover, $\bar{B}^{10}$ is irreducible as $\int_{T_0} w(t) d\mu \gg 0$ and $\alpha(t) \gg 0$, for each $t \in T_0$.

We can now provide the definition of a Cournot-Nash equilibrium of $\Gamma_1$.

**Definition 5.** A strategy selection $\hat{b}^1$ is a Cournot-Nash equilibrium of $\Gamma_1$ if

$$u_t(x(t, \hat{b}^1(t), p(\hat{b}^{10}))) \geq u_t(x(t, \hat{b}^1 \setminus b(t), p(\hat{b}^{10} \setminus b(t)))),$$

for each $b \in B(t)$ and for each $t \in T_1$.

The following argument adapts to our framework the setting of Lemmas 3 and 4 in Sahi and Yao (1989), which we use to prove the existence theorem. Consider an atom $\tau \in T_1$. Given a strategy selection $b^{10}$, let $D$ be a matrix such that $d_{ij} = \bar{D}^{10}_{ij} - b^{10}_{ij}(\tau)\mu(\tau)$, $i, j = 1, \ldots, l$. Then, from (1), we have

$$\sum_{i=1}^l p^i(b^{10})(d_{ij} + b^{10}_{ij}(\tau)\mu(\tau)) = p^i(b^{10})\sum_{i=1}^l (d_{ij} + b^{10}_{ij}(\tau)\mu(\tau)), j = 1, \ldots, l,$$

from which we obtain

$$-\sum_{i=1}^l b^{10}_{ij}(\tau) + \sum_{i=1}^l b^{10}_{ij}(\tau)\frac{p^i(b^{10})}{p^i(b^{10})} = \sum_{i=1}^l \frac{d_{ij}}{\mu(\tau)} - \sum_{i=1}^l \frac{d_{ij} p^i(b^{10})}{p^i(b^{10})}, j = 1, \ldots, l.$$

Then,

$$x^j(\tau, b^{10}(\tau), p(b^{10})) = w^j(\tau) + \sum_{i=1}^l \frac{d_{ij}}{\mu(\tau)} - \sum_{i=1}^l \frac{d_{ij} p^i(b^{10})}{p^i(b^{10})}, j = 1, \ldots, l.$$
from which we obtain

\[
(\mu(\tau))x^j(\tau, b^{10}(\tau), p(b^{10})) = 1 - \frac{\sum_{i=1}^{l} d_{ij}p^i(b^{10})}{p^j(b^{10})}, \quad j = 1, \ldots, l. 
\]

(2)

It is possible to show that Lemmas 3 and 4 in Sahi and Yao (1989) still hold for an atom \( \tau \) when their matrices \( C \) and \( A \) are replaced, respectively, with \( D \) and \( \bar{B}^{10} \), and their Equation (14) is replaced with (2). Lemmas 3 and 4 together imply that the “best response set” of an atom in \( \Gamma_1 \) is nonempty, convex, and compact.

We can now prove the existence of a Cobb-Douglas-Cournot-Nash equilibrium of \( \Gamma \).

**Theorem 1.** Under Assumptions 1, 2, and 3, there exists a Cobb-Douglas-Cournot-Nash equilibrium of \( \Gamma, \hat{b} \).

**Proof.** The assumption that \( \mu(T) < \infty \) implies that \( T_1 \) may be either finite or countably infinite. We shall consider the case where \( T_1 \) contains countably infinite atoms as the argument we use for this case holds, a fortiori, when it contains a finite number of atoms. Let \( \Phi : \prod_{t \in T_1} B(t) \to \prod_{t \in T_1} B(t) \) be a correspondence such that \( \Phi(b^1) = \{ b^1 \in \prod_{t \in T_1} B(t) : b^1(t) \in \Phi_t(b^1), \text{ for each } t \in T_1 \} \) where, for each \( t \in T_1 \), the correspondence \( \Phi_t : \prod_{t \in T_1} B(t) \to B(t) \) is such that \( \Phi_t(b^1) = \arg\max \{ u_t(x(t, b^1 \setminus b(t), p(b^{10} \setminus b(t)))) : b \in B(t) \} \). \( \prod_{t \in T_1} R^2_+ \) is a locally convex Hausdorff space as it is a metric space. \( \prod_{t \in T_1} B(t) \) is a nonempty, convex, and compact subset of \( \prod_{t \in T_1} R^2_+ \) as \( B(t) \) is nonempty, convex, and compact, for each \( t \in T_1 \). Consider a trader \( \tau \in T_1 \). For each \( b^1 \in \prod_{t \in T_1} B(t) \), \( \Phi_\tau(b^1) \) is nonempty, convex, and closed, as Lemma 4 in Sahi and Yao (1989) holds in our framework. Moreover, \( \Phi_\tau \) is upper hemicontinuous by the Berge Maximum Theorem (see Theorem 17.31 in Aliprantis and Border (2006), p. 570). Then, \( \Phi_\tau \) has a closed graph, by the Closed Graph Theorem (see Theorem 17.11 in Aliprantis and Border (2006), p. 561) as \( \prod_{t \in T_1} B(t) \) is nonempty, convex, and compact subset of \( \prod_{t \in T_1} R^2_+ \). But then, the correspondence \( \Phi \) has nonempty, convex values, and a closed graph. Therefore, by the Kakutani-Fan-Glicksberg Theorem (see Theorem 17.55 in Aliprantis and Border (2006), p. 583) there exists a fixed point \( \hat{b}^1 \) of \( \Phi \), which is a Cournot-Nash equilibrium \( \hat{b}^1 \) of \( \Gamma_1 \). Let \( \hat{b} \) be a strategy selection of \( \Gamma \) such that \( \hat{b}(t) = \hat{b}^{10}(t) \), for each \( t \in T \). \( \bar{B} \) is irreducible as \( \bar{B} \) is irreducible. Consider a trader \( \tau \in T_1 \). Then, \( u_\tau(x(\tau, \hat{b}(\tau), p(\hat{b}))) \geq u_\tau(x(\tau, \hat{b} \setminus b(\tau), p(\hat{b} \setminus b(\tau)))) \), for each \( b \in B(\tau) \),
as $\hat{b}^1$ is a Cournot-Nash equilibrium of $\Gamma_1$. Consider a trader $\tau \in T_0$. $x(\tau, \hat{b}(\tau), p(\hat{b})) \in X_{p(\hat{b})}(\tau)$ as $x^j(\tau, \hat{b}(\tau), p(\hat{b})) = \frac{\alpha^j(\tau) \sum_{i=1}^l p^i(\hat{b}) w_i(\tau)}{p^j(\hat{b})}$, $j = 1, \ldots, l$. Suppose that there exists $\bar{b} \in B(\tau)$ such that $u_\tau(x(\tau, \hat{b} \setminus \bar{b}(\tau), p(\hat{b} \setminus \bar{b}(\tau)))) > u_\tau(x(\tau, \hat{b}(\tau), p(\hat{b})))$. It is immediate to verify that $p(\hat{b} \setminus \bar{b}(\tau)) = p(\hat{b})$. Let $\bar{x} = x(\tau, \hat{b} \setminus \bar{b}(\tau), p(\hat{b}))$. Then, it is straightforward so show that $\bar{x} \in \Delta_{p(\hat{b})}(\tau)$. But then, $u_\tau(\bar{x}) > u_\tau(x(\tau, \hat{b}(\tau), p(\hat{b})))$ and $\bar{x} \in \Delta_{p(\hat{b})}(\tau)$, a contradiction. Therefore, $u_\tau(x(t, \hat{b}(t), p(\hat{b}))) \geq u_\tau(x(t, \hat{b} \setminus b(t), p(\hat{b} \setminus b(t))))$, for each $b \in B(t)$ and for each $t \in T$. Hence, $\hat{b}$ is a Cobb-Douglas-Cournot-Nash equilibrium of $\Gamma$.

4 Cournot-Walras equilibrium

In this section, we describe the concept of Cournot-Walras equilibrium proposed by Busetto et al. (2008). The atomless part has Walrasian demands represented by the function $x^0(\cdot, p) : T_0 \to R^l_+$, defined in Section 2. Consider now the atoms’ strategies. Let $e \in R^{l^2}$ be a vector such that $e = (e_{11}, e_{12}, \ldots, e_{l-1,l}, e_{ll})$. A strategy correspondence is a correspondence $E : T_1 \to \mathcal{P}(R^{l^2})$ such that, for each $t \in T_1$, $E(t) = \{e \in R^{l^2} : e_{ij} \geq 0, i, j = 1, \ldots, l; \sum_{j=1}^l e_{ij} \leq w^i(t), i = 1, \ldots, l\}$. A strategy selection is an integrable function $e : T_1 \to R^{l^2}$ such that, for each $t \in T_1$, $e(t) \in E(t)$. For each $t \in T_1$, $e_{ij}(t)$, $i, j = 1, \ldots, l$, represents the amount of commodity $i$ that trader $t$ offers in exchange for commodity $j$. We denote by $e \setminus e(t)$ a strategy selection obtained by replacing $e(t)$ in $e$ with $e \in E(t)$. With a slight abuse of notation, $e \setminus e(t)$ will also denote the value of the strategy selection $e \setminus e(t)$ at $t$. Given a strategy selection $e$, consider the following equation:

$$\int_{T_0} x^0(t, p) d\mu + \sum_{i=1}^l \int_{T_1} e_{ij}(t) d\mu \frac{p^j}{p^i} = \int_{T_0} w^j(t) d\mu + \sum_{i=1}^l \int_{T_1} e_{ji}(t) d\mu, \quad (3)$$

$j = 1, \ldots, l$.

When defining their notion of Cournot-Walras equilibrium, Busetto et al. (2008) assumed the existence of a market clearing price vector satisfying (3). Moreover, as market clearing price vectors may not be unique, they had to define their equilibrium concept with respect to an arbitrary selection among market clearing prices. The following proposition shows that, thanks
to the assumption that traders belonging to the atomless part have Cobb-
Douglas preferences, there exists a unique, up to a scalar multiple, price
vector \( p \in R^l_+ \) which satisfies Equation (3).

**Proposition 3.** Under Assumptions 1, 2, and 3, for each strategy selection \( e \), that there exists a unique, up to a scalar multiple, price vector \( p \in R^l_+ \) which satisfies Equation (3).

**Proof.** Consider a strategy selection \( e \). Let \( e^{10} : T \to R^l_+ \) be a function such that \( e^{10}(t) = e(t) \), for each \( t \in T_1 \), and \( e^{10}(t) = b^0(t) \), for each \( t \in T_0 \). Then, \( e^{10} \) is integrable by the same argument used for the function \( b^{10} \). Define the aggregate matrix \( \bar{E}^{10} = (\int_{T_0} e^{10}_{ij}(t) \, d\mu) \). \( \bar{E}^{10} \) is irreducible by the same argument used for the matrix \( \bar{B}^{10} \). (3) can be written as

\[
\sum_{i=1}^l p^j \left( \int_{T_0} w^i(t) \alpha^j(t) \, d\mu + \int_{T_1} e_{ij}(t) \, d\mu \right) = p^j \left( \sum_{i=1}^l \bar{e}^{10}_{ij} \right) + \sum_{j=1}^l \int_{T_1} e_{ji}(t) \, d\mu, \quad j = 1, \ldots, l.
\]

Then, (3) can be rewritten as

\[
\sum_{i=1}^l p^j \bar{e}^{10}_{ij} = p^j \left( \sum_{i=1}^l \bar{e}^{10}_{ij} \right), \quad j = 1, \ldots, l. \tag{4}
\]

By Lemma 1 in Sahi and Yao (1989), there is a unique, up to a scalar multiple, price vector \( p \in R^l_+ \) satisfying (4) as \( \bar{E}^{10} \) is irreducible. Hence, there exists a unique, up to a scalar multiple, price vector \( p \in R^l_+ \) which satisfies Equation (3).

We denote by \( p(e) \) a function which associates, with each strategy selection \( e \), the unique, up to a scalar multiple, price vector \( p \) satisfying (3). It is straightforward to verify that \( p(e') = p(e'') \) if \( \int_{T_1} e'(t) \, d\mu = \int_{T_1} e''(t) \, d\mu \). For each strategy selection \( e \), let \( x^1(\cdot, e(\cdot), p(e)) : T_1 \to R^l_+ \) denote a function such that

\[
x^{1j}(t, e(t), p(e)) = w^j(t) - \sum_{i=1}^l e_{ji}(t) + \sum_{i=1}^l e_{ij}(t) \frac{p^j(e)}{p^j(e)}, \quad j = 1, \ldots, l, \text{ for each } t \in T_1.
\]
Equation (3), it is straightforward to show that the function $x(t)$ such that $x(t) = x^1(t, e(t), p(e))$, for all $t \in T_1$, and $x(t) = x^0(t, p(e))$, for all $t \in T_0$, is an allocation.

At this stage, we are able to define the concept of Cournot-Walras equilibrium.

**Definition 6.** A pair $(\bar{\hat{e}}, \tilde{x})$, consisting of a strategy selection $\bar{\hat{e}}$ and an allocation $\tilde{x}$ such that $\tilde{x}(t) = x^1(t, \hat{e}(t), p(\hat{e}))$, for each $t \in T_1$, and $\tilde{x}(t) = x^0(t, \hat{e}(t), p(\hat{e}))$, for each $t \in T_0$, is a Cournot-Walras equilibrium of $\mathcal{E}$ if

$$u_t(x^1(t, \bar{\hat{e}}(t), p(\bar{\hat{e}}))) \geq u_t(x^1(t, \bar{\hat{e}} \setminus e(t), p(\bar{\hat{e}} \setminus e(t)))),$$

for each $e \in E(t)$ and for each $t \in T_1$.

## 5 Cobb-Douglas-Cournot-Nash and Cournot-Walras equilibrium

Since either the bids of the traders belonging to the atomless part, at a Cobb-Douglas-Cournot-Nash equilibrium, and their Walrasian demands, at a Cournot-Walras equilibrium, depend on the the parameters of their Cobb-Douglas utility functions, we are lead to raise the question whether these two equilibrium notions are equivalent in our framework. The following theorem provides a positive answer to our question. Indeed, it shows that the set of the Cobb-Douglas-Cournot-Nash equilibrium allocations coincides with set of the Cournot-Walras equilibrium allocations.

**Theorem 2.** Under Assumptions 1, 2, and 3, (i) if $\tilde{\hat{b}}$ is a Cobb-Douglas-Cournot-Nash equilibrium of $\Gamma$, then there is a strategy selection $\bar{\hat{e}}$ such that the pair $(\bar{\hat{e}}, \tilde{x})$, where $\tilde{x}(t) = x^1(t, \hat{e}(t), p(\hat{e})) = x^1(t, \bar{\hat{e}}(t), p^{10}(\bar{\hat{e}}))$, for each $t \in T_1$, and $\tilde{x}(t) = x^0(t, \hat{e}(t), p(\hat{e})) = x^0(t, p^{10}(\bar{\hat{e}}))$, for each $t \in T_0$, is a Cournot-Walras equilibrium of $\mathcal{E}$; (ii) if $(\bar{\hat{e}}, \tilde{x})$ is a Cournot-Walras equilibrium of $\mathcal{E}$, then there is a Cobb-Douglas-Cournot-Nash equilibrium $\tilde{\hat{b}}$ of $\Gamma$ such that $\tilde{x}(t) = x(t, \tilde{\hat{b}}(t), p(\tilde{\hat{b}}))$, for each $t \in T$.

**Proof.** (i) Let $\tilde{\hat{b}}$ be a Cobb-Douglas-Cournot-Nash equilibrium of $\Gamma$. Let $\bar{\hat{e}}$ be a strategy selection such that $\hat{e}(t) = \bar{\hat{b}}(t)$, for each $t \in T_1$. Then, $p(\bar{\hat{e}}) = p(\tilde{\hat{b}}) = \tilde{E}^{10} = \tilde{B}$ and $p(\tilde{\hat{b}})$ satisfies Equation (1). But then, it is straightforward to verify that $x(t, \tilde{\hat{b}}(t), p(\tilde{\hat{b}})) = x^1(t, \hat{e}(t), p(\hat{e}))$, for each $t \in T_1$, and $x(t, \tilde{\hat{b}}(t), p(\tilde{\hat{b}})) = x^0(t, p(\bar{\hat{e}}))$, each $t \in T_0$. Suppose that there is a trader $\tau \in T_1$ and a strategy $\bar{e} \in E(\tau)$ such that $u_\tau(x^1(\tau, \bar{e}(\tau), p(\bar{e} \setminus \bar{e}(\tau)))) >$
Conclusion

We proved the existence of a Cobb-Douglas-Cournot-Nash equilibrium. Under Assumptions 1, 2, and 3, there exists a Cournot-Walras equilibrium of \( \mathcal{E} \), \((\hat{e}, \tilde{x})\).

The following corollary is a straightforward consequence of Theorem 2.

**Corollary.** Under Assumptions 1, 2, and 3, there exists a Cournot-Walras equilibrium of \( \mathcal{E} \), \((\hat{e}, \tilde{x})\).

### 6 Conclusion

In this paper, we reconsidered two models of noncooperative oligopoly in general equilibrium which are both a reformulation of a particular strategic market game, the so called Shapley’s window model, introduced by Busetto et al. (2008), (2011). The novelty introduced in this paper is the assumption that the preferences of the traders belonging to the atomless part are represented by Cobb-Douglas utility functions. Two kind of results are obtained. First, we proved the existence of a Cobb-Douglas-Cournot-Nash equilibrium. Second, we showed the equivalence between the set of the Cobb-Douglas-Cournot-Nash equilibrium allocations and the set of the Cournot-Walras equilibrium allocations.

Busetto et al. (2012) have investigated the limit relationship between the Cournot-Nash and the Walras equilibrium. They partially replicated
the exchange economy by increasing the number of atoms without affecting the atomless part while ensuring that the measure space of agents remains finite. Then, they showed that any sequence of Cournot-Nash equilibrium allocations of the strategic market game associated with the partially replicated exchange economies approximates a Walras equilibrium allocation of the original exchange economy. A next step of our analysis could be to undertake a similar investigation in the framework of this paper.

References


